Computational evidence of an approximate splines solution of o.d.e.

Praciano-Pereira, T *

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tarcisio@member.ams.org
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Editor Tarcisio Praciano-Pereira
tarcisio@member.ams.org

Resumo

Neste trabalho estou mostrando a evidência computacional da aproximação spline na solução de uma equação diferencial.

Trata-se de um exemplo, uma equação de fácil solução, $y' = y$ mas que eu escolhi, tanto porque é de fácil solução, conhecemos a solução exata, mas também por sua grande variação, a derivada da solução cresce exponencialmente. Então o objetivo do artigo é a descrição do método.

palavras chave: solução aproximada de e.d.o. aproximação polinomial, operador diferencial.

The aim of this paper is to show the evidence of the good approximation of a spline solution of ordinary differential equation. The example used is of a simple equation on purpose, as the solution is easy to obtain, and it will be used as comparison, but not only for this, it is not of bounded variation as the derivative has exponential growth.

Thus, the goal is a description of the method.

keywords: differential operator, splines base, splines kernel.

*tarcisio@member.ams.org

1 How the program has been used

If the abstract seams clear you may safely skip this introduction and go over to the next section where the program is described.

Suppose we have a family of $n+1$ splines with compact support of arbitrary class of differentiation $\mathcal{C}^m([a,b]$ each one expanding over two successive sub-intervals of the the uniform partition $\Pi_n([a,b])$ and in fact we have them, see [3], and an easy way to understand how they have been build is the figure (1) page 1, which has been obtained by convolving a square kernel with support

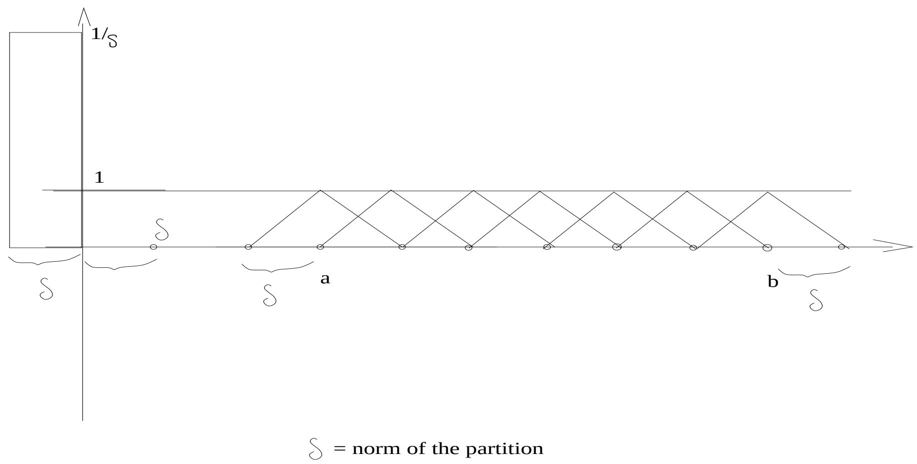

Figura 1:

$[-\delta, 0]$ with the characteristic functions of the sub-intervals of $\Pi_n([a,b])$, under the assumption that δ is common measure of these sub-intervals. As the characteristic functions are a quasi-partition of the unity on $[a,b]$ then the functions obtained this way form a continuous partition of the unity. If the multiplying kernel is of class $\mathcal{C}^m([a,b])$ and still have a support with measure δ, then we have the family we need.

The reader will readily note that this restriction of the measure of the support of the kernel is not necessary, given any kernel, a function whose integral is one, and with some kind of regularity, let's say continuous, it will regularize this quasi-partition of the unity into a continuous partition of the unity. But having this measure will make the things simpler at the program later because the picture will be almost the one we have in figure (1) page 1, but having bell shaped kernels translations, instead of triangles, hence their sum turns out to be the sum of two successive kernel translations.

This kernel of arbitrary class of differentiability has been constructed in [3] as the m-th convolution power of the characteristic function of the interval $[0,1]$. In fact there we have a kernel with support $[0,m]$, the convolution power of $\chi_{[0,1]}$, but if q is one such power, then $\rho(x) = wq(wx); w = \frac{m}{\delta}$; is a kernel whose support measures δ.

Let us call ρ the this kernel, it class of continuity being $\mathcal{C}^m([a,b])$, and let it be $\eta = \rho * \chi_{[0,\delta]}$ its convolution with the characteristic function of the interval

$[0, \delta]$, where δ is the common measure of the subintervals in $\Pi_n([a, b])$.

We can easily show that the functions in the family are the translations of this convolution to each node x_k of the partition $\Pi_n([a, b])$ this enables us to set the following definitions:

$$\eta_k(x) = \eta(x - x_k); x_k \text{ a node of } \Pi_n([a, b]); \tag{1}$$

$$\Phi(f) = \sum_{k=0}^{n} f(x_k)\eta_k; \Phi : \mathcal{C}^m([a, b]) \to Spl_n([a, b]); \tag{2}$$

$$I_k \text{ will name the k-th interval of } \Pi_n([a, b]); \tag{3}$$

Φ defined in the equation (eq. 2) is an interpolation projector, see [1].

The summation at equation (eq. 2) turns to be a sum of exactly two items for each value of x because the atoms in the definition of Φ have support of measure 2δ by construction. We are calling η_k or η an atom.

Now we are dealing with the differential equation $y' = y$ then we are going to apply Φ writing

$$\Phi(f') = \sum_{k=0}^{n} f'(x_k)\eta_k; \tag{4}$$

and integrate this derivative with a prescribed initial condition and calculate the norm of the difference between this primitive and the solution that we already know. This will show the precision of the approximate solution.

There are some points here to make the program flow rapidly. First the atoms in equation (eq. 2) are simply translation of a fixed one hence there is only one integration to be calculated no matter what n may be, secondly, our choice of the multiplying kernel, also the factor which produces the atoms, having support measuring δ, makes the sum in equation (eq. 4) turns out to be the sum of two successive atoms for each x, again, no matter what n is. As we are going to use a kernel calculated algebraically the error will remain on the errors to calculate the constants which define this kernel, see [3], and the precision in the calculus of these constants can be improved, finally, there is only one integral to be calculated as the atoms are translations of a given one, this is an exact integral as we are dealing with polynomial splines.

2 How the program works

This section is written with the program in mind. To make the things simpler, the program is produced to work with a partition of $[0, n]$ where the nodes are the integers of this interval. The translation to a uniform partition of is straightforward. But we shall make the explanation if we were at the general case, we mean, if we were working already at the translation to a particular interval $[a, b]$ so we can make reference to the previous equations. The comments at the programs will make this clear to the reader.

In the sum at equation (eq. 2) the atom η_k is the convolution of χ_{I_k} with the kernel ρ; $\eta_k = \rho * \chi_{I_k}$ and we are going to simply this notation writing

$$\eta_k = \rho * \chi_k = \eta(x-k);\ \eta = \rho * \chi_{I_0} \tag{5}$$

This convolution is easy to calculate as

$$(\rho * \chi_k)' = \rho * (\chi_k)' = \rho * (\delta_{x_k} - \delta_{x_{k+1}}) = \rho_{x_k} - \rho_{x_{k+1}}; \tag{6}$$

where in the next-to-last expression in equation (6) we read the difference of the Dirac measure at the end points of $[x_k, x_{k+1}] = I_k$ and, as the measure of the support of ρ is δ, the difference in the last expression is of two functions of disjoint support, or at most intersecting at a boundary point. The integral of this difference over $[x_{k-1}, x_{k+1}]$ is η_k, the atom in equation (eq. 2).

As the support of ρ is an interval whose measure is the norm of the partition, then η_k expands over two subintervals of the partition making the sum in equation (eq. 3) reduced to exactly two parcels. When we have found at which sub-interval x belongs to at the program then $\Phi(f)(x)$ is the summation of two successive translations of η to the end points of $I_k \ni x$. This is exactly the picture (1) page 1 now having bell shapped splines atoms with the same class of differentiability of ρ.

To solve the simple equation we have in mind in this paper all we have to is to integrate equation (eq. 3). which is a sum of translates of η so all we have to do is calculate the integral of $f'(x_k)\eta$ with the initial condition $\int_{I_{k-1}} f'(x_{k-1})\eta_k(t)dt$. But these integrals are one and the same integral of η multiplied by $f'(x_{k-1})$ and is this which makes the things simple:

$$\int_{I_{k-1}} \eta_k(t)dt = \int_{I_0} \eta(t)dt = \int_0^\delta \eta(t)dt = \tag{7}$$

$$= \int_0^\delta \rho * \chi_0(t)dt = \int_0^\delta \int_0^\delta \rho(t) * \chi_0(x-t)dxdt = \int_0^\delta \int_{-t}^{\delta-t} \rho(t) * \chi_0(y)dydt = \delta \tag{8}$$

In this paper, by lack of time, we have calculated the norm $\mathcal{L}^1$ norm, $||f - f'||_1$ and to do this we have modified the module which makes the graphic to calculate the integral of $|f - f'|$. In this version we are using Riemann sums to approximate the integrals but in the future we are going to use a better approach using a kind of polynomial approximation in each interval whose integral will be evaluated. The calculus of the norm of Besov type $(1,1)$ is not difficult and we have already all the machinery appropriate to perform the calculus, it is the same module which makes the graphic, but it demands some extra time to code. This will be done in a near future.

The following two graphics are a preview of what has been already done, they show with two different steps, $\delta \in \{1, 0.1\}$ the graphics of $f(x) = exp(x)$ and its derivative but the graphic of the derivative is drawn with help of the projector in equation (eq. 2) . We have used a very small pass to obtain these graphics and the values of δ show graphically the gain in precision when changing between these two values as the measure of the support of the atom in equation (eq. 2) . A figura (2) página 4,

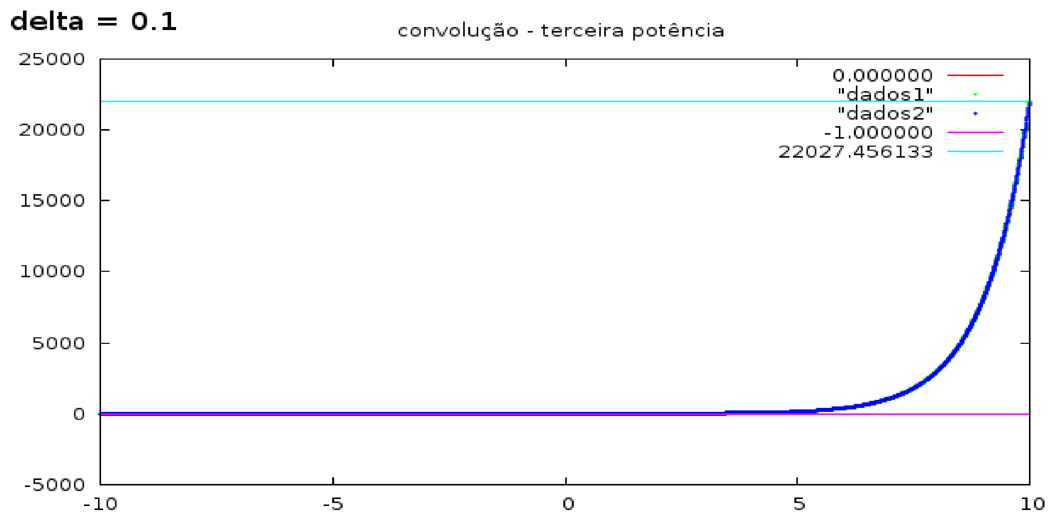

Figura 2: $\delta = 0.1$

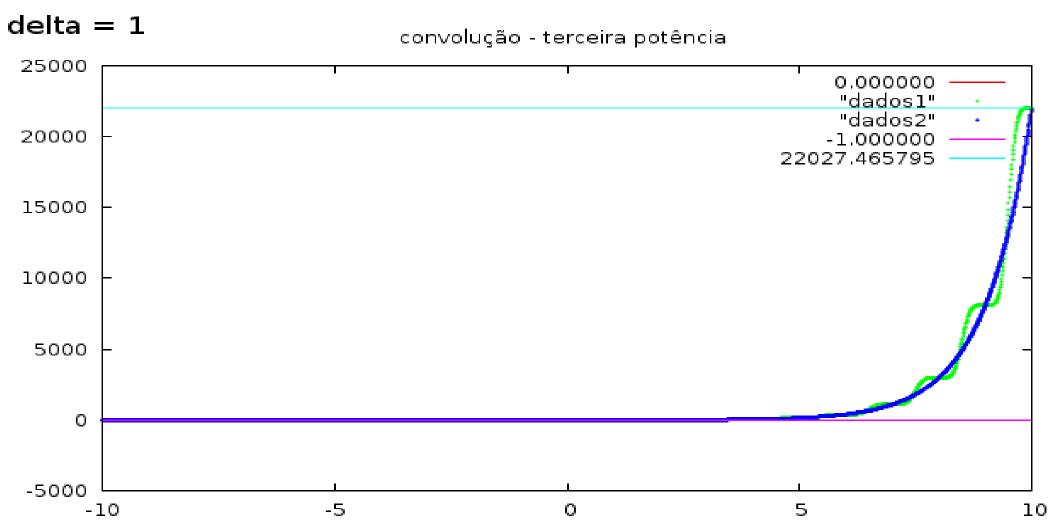

Figura 3: $\delta = 1$

We have not done the job when the partition is not uniform but in that case, again, the integral will be only the result of a "change of variable" in the integral of η and all we have to see is if we still have a good precision as result.

The python programs will be published in a forthcoming paper under GPL, [?].

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