An interpolation projector associated to a non uniform partition

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Abstract

In this paper we present an interpolation projector by modifying a previous construction of a *compact support convolution splines partition of the unity* to produce an interpolation projector without the restriction that the partition of the interval be uniform.

We have been able also to substitute convolution by translations and dilation (the techniques of wavelet).

Keywords: approximate identity, convolution regularization, convolution splines, non uniform partition, partition of the unity, interpolation projectors.

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1 Introduction

This paper deals with a class of splines which are of compact support and I will start making some definitions which do not contradict the usual definitions but add to more elegance to paper if the reader accepts them without constraints.

Definition 1 *0-splines*
0-*splines are linear combination of characteristic functions of intervals.*

This is a simple addition to the usual definition and the reader will see in a few lines more that this closes a gap in class of splines and enhances the use of convolution.

One such linear combination need not to be continuous but it agrees nicely with differentiability condition one should expect provided we do not try to say they are of class "$0-1$" (and we are not), one unity less that the dimension. I shall only say that 0-splines are not necessarily continuous.

Definition 2 *Kernel-splines*
Is the characteristic function of an interval of measure 1 or a convolution power of one such characteristic function or a dilation of any of the previous.

Again here, the lowest level class of splines is made of non continuous funtions, the characteristic functions of intervals.

This definition is biased to the needs of this paper. By kernel one understand a positive function whose integral is one and we can easily create a positive splines function, of compact support and whose integral is one, which is not the convolution power of a characteristic function of an interval of measure one. Take to triangle functions having integral one, they are not, clearly, convolution of a characteristic function of an interval of measure one.

So I have defined as kernel something which is a particular type of kernel.

Definition 3 *Splines with compact support*
Is a linear combination of characteristic functions or the convolution of one such linear combination by a kernel-splines.

O-splines are included in this definition.

I have not been able characterize all splines of compact support starting from this simple atom represented by the kernel splines. This seams to be a nice question.

2 Partition of the unity associated to a non-uniform partition

Let me consider in this section a partition of the interval $[a, b]$ which is not uniform.

Let me define

Definition 4 *Deficit of a partition*

Is the least length of the sub-intervals in the partition.

So we have a finite strictly increasing succession of nodes

$$a = x_1, \ldots, x_k, \ldots, x_{n-1} = b \quad (1)$$

$$\Delta x \text{ is the deficit of the partition} \quad (2)$$

$$x_0 = a - \Delta x \; ; \; x_n = b + \Delta x \quad (3)$$

$$(4)$$

that is, I am imbedding the partition of $[a, b]$ in a bigger one of the interval $[x_0, x_n]$ with the same deficit.

Now I will define a set of 1-splines of compact support in the following way

$$\text{for k=0 to n-2} \quad (5)$$

$$f_n \text{ is the triangle function determined } by \quad (6)$$

$$(x_k, 0), \; (x_{k+1}, 1), \; (x_{k+1}, 0) \quad (7)$$

$$(8)$$

that is, *for each three nodes in sequence*, I will draw a straight line starting from the x-axis, at the point x_k, to the line $y = 1$ above the next point and again a straight line starting from this point on $y = 1$ back to the next node on the x-axis. The algebraic (formal) definitions of these equations are important but this geometric explanation together with the picture (1) page 3, is much easier to understand. I will not have a direct use of formal definition of these equations in the rest of the paper, and do not forget these are compact support 1-splines each having as support two consecutive sub-intervals of the partition.

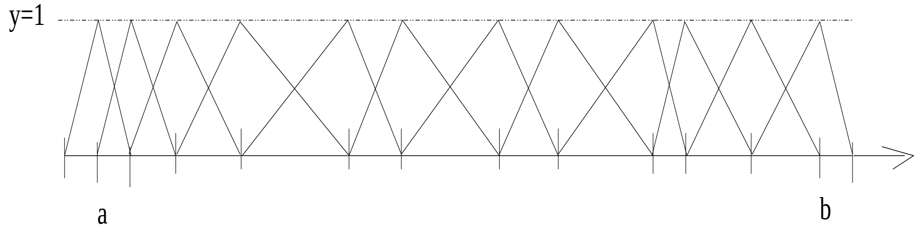

Figure 1: A set of 1-splines

Theorem **1** *$\mathcal{C}^0$ partition of the unity The sum all over the function f_n is one on $[a, b]$ collapsing to zero outside the interval $[x_0, x_n]$ hence a $\mathcal{C}^0$ finite partition of the unity subordinated to an open cover of $[a, b]$.*

Dem: *As the sum of two function of first degree is a function of degree less or equal to one, thus its graphic is a straight line. For each pair of consecutive straight lines the end points are alternatively on $y = 1$ or $y = 0$ then sum will be over the line $y = 1$.*

q.e.d .

If we consider now the points

$$(x_k, y_k) \ ; \ k = 1 \cdots n - 1 \tag{9}$$

the functions

$$\sum_{k=1}^{n-1} y_k f_k \tag{10}$$

is the linear interpolation of the points in equation (9). The proof of this fact may easily done with geometric considerations similar to those we have used in the proof of the previous theorem, sum of straight lines is a straight line again.

This also proves the theorem

Theorem **2** *Basis of Spl-1($[x_0, x_n]$)*

The set of functions $(f_k)_{k=1}^{n-1}$ is a basis for the vector space of 1-splines of compact support $[x_0, x_n]$.

We have, thus, an interpolation projector induced by this partition of the unity.

Theorem **3** *Interpolation projector*

The operator defined on $\mathcal{C}^n_c([x_0, x_n])$ *by*

$$\mathcal{P}(f) = \sum_{k=1}^{n-1} f(x_k) f_k \tag{11}$$

is a projector of the space $\mathcal{C}^n([a, b])$ *onto Spl-1*$([x_0, x_n])$

Dem:

$\mathcal{P}$ is linear operator, of order 2, hence a projector, which has the identity as fixed point (which is a consequence that is generated by a partition of the unity.

q.e.d .

3 An interpolation projector

In this section I shall prove the following result:

Theorem **4** *Interpolation projector associated to non-uniform partitions*

Let Π *be a partition of* $[a, b]$ *as in the previous section with the two external nodes* $x_0 = a - \Delta x, x_n = b + \Delta x$*, where* Δ *is the deficit of the partition* Π.

There is finite $\mathcal{C}^n$*-partition of the unity subordinated to an open cover* $[a, b]$ *which is the basis of the space Spl-n*$([x_0, x_n])$ *of all* n *splines with compact support* $[x_0, x_n]$ *and an interpolation projector of a space of continuous functions with compact support* $[x_0, x_n]$ *onto Spl-n*$([x_0, x_n])$*. There is, moreover an element of Spl-n*$([x_0, x_n])$ *which is interpolates the points*

$$(x_1, y_1), \cdots, (x_{n-1}, y_{n-1})$$

I need the lemma

Lemma **1** *There is* $\mathcal{C}^n$*-splines kernel with support* $[-\frac{\Delta x}{4}, \frac{\Delta x}{4}]$

Dem:

Call η *the* $n+1$*th convolution power of* $\chi[-\frac{1}{2}, \frac{1}{2}]$ *is* $\mathcal{C}^n$*-splines, but the supp*(η) *will be* $[-\frac{n}{2}, \frac{n}{2}]$ *as each convolution adds together the supports.*

Define ρ *such that* $\frac{n\rho}{2} = \frac{\Delta x}{4}$.

The dilation of η *by* $r = \frac{1}{\rho}$

$$\eta_\rho(x) = r\eta(rx)$$

is the desired kernel-splines. **q.e.d .**

Now consider the $\mathcal{C}^0$-partition of the unity defined in the (6) .

The convolution of f_k with the splines-kernel η_ρ produces the family ϕ_k.

Lemma **2** *The family $\phi_k = \eta_\rho * f_k$ is a $\mathcal{C}^n$-partition of the unity for the interval $[a, b]$*

Dem: *The class of differentiability is well know consequence of regularization by convolution. As the function f_k adds to 1 on $[a, b]$ then $\eta_\rho * f_k$ will to sum up to 1 on $[a, b]$.* **q.e.d .**

Lemma **3** *The splines $\eta_\rho * f_k$ are linearly independent*

Dem:

Because convolution and translation commutes, then $\eta_\rho * f_k$ are translation of a fixed one splines among this family of splines.

q.e.d .

These lemmas proves that the family $\phi_k = \eta_\rho * f_k$ spans a vector space of splines with compact support $[x_0, x_n]$ hence they defines in the same manner of theorem an interpolation projector of a space (this is a very interesting point) of any space of continuous function over Spl-$n([x_0, x_n])$.

Given the points

$$(x_1, y_1), \cdots, (x_{n-1}, y_{n-1})$$

we have an element of Spl-$n([x_0, x_n])$ which is

$$\sum_{k=1}^{n-1} y_k \phi_k$$

is equal to y_k on x_k because there is exactly one function of the family $(\phi_k)_{k=1}^{n-1}$ which is non-zero on x_k because the support of η_ρ measures $\frac{\Delta x}{2}$ and the deficit of the partition is Δx, hence these are the precision points of this interpolation.

4 Final remarks

A simple but powerful result is that all the material in this paper can be easily extended to non-polynomial splines. Take kernel η as an arbitrary one of class $\mathcal{C}^\infty$ for example.

The last theorem also has an indefinite expression regarding the domain of the interpolation operators. This open the question to many possible results (or to new proofs of old results) of approximation of arbitrary continuous or differentiable functions by splines.

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This paper, and research beneath it could not have been possible if I did not had all the programs I have in my Linux box and I feel very pleased to thank the Debian foundation and all free software developers which gave me this opportunity, this is really crucial in an undeveloped country as Brazil.

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Harold S. Shapiro

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